Finitary Definitions of Multicategory

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Genova

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joint work with John Bourke

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Introduction

Monoidal Cats $(\mathbb{C}, \otimes, \dots)$

Pros: Finitary and infinitary, *nice* definition

Cons: tensor less direct (colim)

Closed Cats $(\mathbb{C}, [-,-], \dots)$

Pros: Finitary, homs are more explicit (lim)

Cons: Iterated homs

Multicats

Pros: use only universal properties, no lim/colim

Cons: Only infinitary...

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Classical Multicategories

Classical Definition

A **multicategory** $\mathcal{C}$ consists of:

- a collection of **objects**
- for each (possibly empty) list $a_1, \ldots, a_n$ of objects and each object b, a set $\mathcal{C}_n(a_1, \ldots, a_n; b)$, or $\mathcal{C}_n(\bar{a}; b)$, of **n-multimaps**
- for each object a an element $1_a \in \mathcal{C}_1(a; a)$
- **substitution operations** (with $K = \sum_{i=1}^{n} k_i$)

$$\mathcal{C}_n(b_1, \ldots, b_n; c) \times \prod_{i=1}^{n} \mathcal{C}_{k_i}(\bar{a}_i; b_i) \to \mathcal{C}_K(\bar{a}_1, .., \bar{a}_n; c)$$

$$(g, f_1, \ldots, f_n) \mapsto g(f_1, \ldots, f_n)$$

We also require *associativity conditions* and *identity laws*.

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$$h \circ_i (g \circ_j f) = (h \circ_i g) \circ_{j+i-1} f \quad \text{for} \quad 1 \leq i \leq m, 1 \leq j \leq n \tag{1}$$

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(Note: If we have $\circ_i$ then, $g(f_1, ..., f_n) := (...((g \circ_1 f_1) \circ_{k_1+1} f_2)...) \circ_{K-k_n+1} f_n$)

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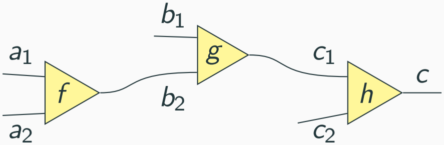
Example of (1)

h, g and f binary maps, $i = 1$ and $j = 2$

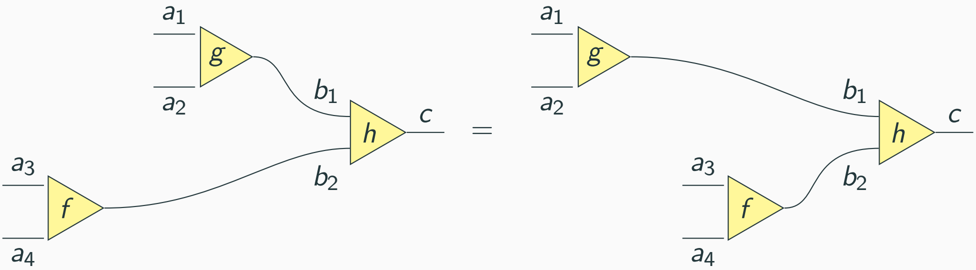
$$(h \circ_1 g) \circ_2 f = h \circ_1 (g \circ_2 f)$$



Example of (2)

h, g and f binary maps, $i = 1$ and $j = 2$

$$(h \circ_1 g) \circ_3 f = (h \circ_2 f) \circ_1 g$$



Finitary Multicategories

Definition

A **finitary multicategory** consists of a category $\mathbb{C}$ together with:

- For $n \leq 4$ a profunctor $\mathcal{C}_n(-;-) : (\mathbb{C}^n)^{op} \times \mathbb{C} \to \mathbf{Set}$

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Left Representability

n-**ary map classifier** for $\bar{a}$ consists of a representation of $\mathcal{C}_n(\bar{a}; -) : \mathcal{C} \to$ **Set**

i.e. a (universal) multimap $\theta_a \colon a_1, \ldots, a_n \to m(a_1, \ldots, a_n)$ inducing bijections

$$- \circ \theta_a : \mathcal{C}_1(m(a_1, \ldots, a_n); b) \to \mathcal{C}_n(a_1, \ldots, a_n; b)$$

a universal multimap is said to be **left representable** if it induces bijections

$$- \circ_1 \theta_a \colon \mathcal{C}_{1+k}(m(a_1, \ldots, a_n), \bar{x}; b) \to \mathcal{C}_{n+k}(a_1, \ldots, a_n, \bar{x}; b)$$

Definition

A finitary multicategory $\mathcal{C}$ is said to be left representable if it admits left universal nullary and binary map classifiers u and $\theta_{a,b}$.

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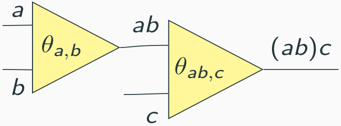
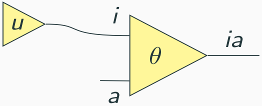
One proves that the following multimaps are 3-ary and unary map classifier:





Aim: construct a functor $K\colon \mathbf{FMult}_I \to \mathbf{Skew}_I$.

$$\alpha\colon (ab)c \to a(bc) \in \mathbb{C}((ab)c, a(bc)) = \mathcal{C}_1((ab)c; a(bc)) \cong \mathcal{C}_3(a,b,c; a(bc))$$

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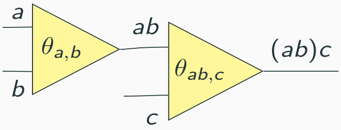

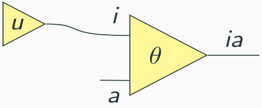

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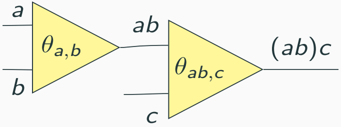

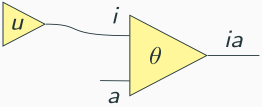

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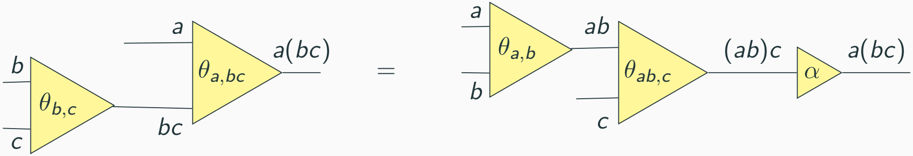
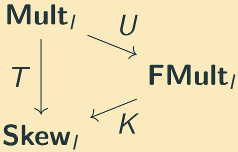


Theorem (Bourke & Lobbia)

There is an equivalence $U\colon \mathbf{Mult}_I \to \mathbf{FMult}_I$ *fitting in the diagram*

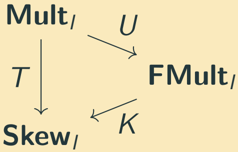


Proof (Sketch).

1. Prove that K is fully faithful (using a characterisation of morphisms in $\mathbf{FMult}_I$).
2. T is an equivalence (Bourke & Lack), hence K is essentially surjective.
3. U is an equivalence as well. □

Theorem (Bourke & Lobbia)

There is an equivalence $U\colon \mathbf{Mult}_l \to \mathbf{FMult}_l$ *fitting in the diagram*



Proof (Sketch).

1. Prove that K is fully faithful (using a characterisation of morphisms in $\mathbf{FMult}_l$).
2. T is an equivalence (Bourke & Lack), hence K is essentially surjective.
3. U is an equivalence as well. □

Left Representable Closed

A finitary multicategory is (left) **closed** if there exists an object $[b, c]$ and binary map $e_{b,c} \colon [b, c], b \to c$ inducing isomorphisms (for $n = 0, 1, 2, 3$)

$$e_{b,c} \circ_1 - \colon \mathcal{C}_n(\overline{x}; [b, c]) \to \mathcal{C}_{n+1}(\overline{x}, b; c)$$

Theorem (Bourke & Lobbia)

The equivalence $K \colon \mathbf{FMult}_l \to \mathbf{Skew}_l$ induces another one $K_c \colon \mathbf{FMult}_l^{cl} \to \mathbf{Skew}_l^{cl}$ between finitary left representable closed multicategories and left normal skew closed monoidal categories.

Remark: There is also an equivalence $\mathbf{FMult}^{cl} \to \mathbf{Closed}$ between closed finitary multicategories with unit and *closed categories*.

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Finitary Skew Multicategories

Skew Multicategories

A **skew multicategory** (Bourke & Lack, J.Alg., 2018) $\mathcal{C}$ consists of

- a set of objects $\mathcal{C}_0$
- for each $x \in \mathcal{C}_0$ a set $\mathcal{C}_0^l(\ ; x)$ of **nullary maps**
- for each $\bar{a}, b \in \mathcal{C}_0$ two sets $\mathcal{C}_n^t(\bar{a}; b)$ and $\mathcal{C}_n^l(\bar{a}; b)$ of **tight/loose n-multimaps**
- for each $n > 0$, $\bar{a}, b \in \mathcal{C}_0$ a function $j_{\bar{a},b} \colon \mathcal{C}_n^t(\bar{a}; b) \to \mathcal{C}_n^l(\bar{a}; b)$
  ("inclusion" of tight multimaps into loose ones)
- for each $x \in \mathcal{C}$ a tight multimap $1_x \in \mathcal{C}_1^t(x; x)$
- **substitution operation** $g(f_1, \ldots, f_n)$, which is tight just when g and f_1 are.

+ associativity and unit axioms.

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Skew Multicategories

A **skew multicategory** (Bourke & Lack, J.Alg., 2018) $\mathcal{C}$ consists of

- a set of objects $\mathcal{C}_0$ **E.g.** Categories with a choice of finite products
- for each $x \in \mathcal{C}_0$ a set $\mathcal{C}_0^l(\ ; x)$ of **nullary maps** **E.g.** objects of $\mathbb{A}$
- for each $\bar{a}, b \in \mathcal{C}_0$ two sets $\mathcal{C}_n^t(\bar{a}; b)$ and $\mathcal{C}_n^l(\bar{a}; b)$ of **tight/loose n-multimaps**
  E.g. $F : \mathbb{A}_1 \times \ldots \times \mathbb{A}_n \to \mathbb{B}$ preserving products up-to-iso, tight if *strictly* in $\mathbb{A}_1$
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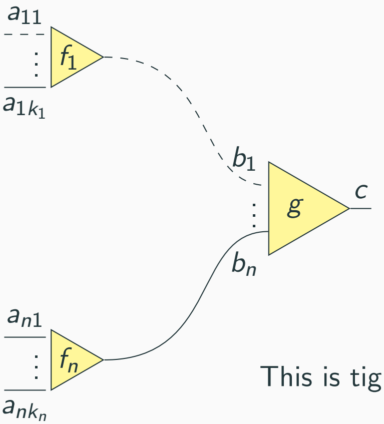
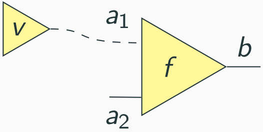
Tight maps

Tight maps are understood as maps "strict" in the first variable.



This is tight



This is loose

Finitary Version

A **finitary skew multicategory** consists of a category $\mathbb{C}$ together with:

- For $1 \leq n \leq 4$ a profunctor $\mathcal{C}^t_n(-;-) : (\mathbb{C}^n)^{op} \times \mathbb{C} \to \mathbf{Set}$

  such that, when $n = 1$, we have $\mathcal{C}^t_1(-;-) = \mathbb{C}(-,-) : \mathbb{C}^{op} \times \mathbb{C} \to \mathbf{Set}$
- For $\mathbf{n = 0, 1, 2}$ an additional profunctor $\mathcal{C}^l_n(-;-) : (\mathbb{C}^n)^{op} \times \mathbb{C} \to \mathbf{Set}$
- For $n = 1, 2$ natural transformations $j_n : \mathcal{C}^t_n(-;-) \to \mathcal{C}^l_n(-;-)$
- (natural) substitution functions

$$\circ_i : \mathcal{C}^x_n(\bar{b}; c) \times \mathcal{C}^y_m(\bar{a}; b_i) \to \mathcal{C}^{x \circ_i y}_{n+m-1}(b_{<i}, a, b_{>i}; c)$$

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Theorem (Bourke & Lobbia)

There are triangles of equivalences (T_l and T^{cl} already defined by Bourke & Lack)

$$\begin{array}{ccc} \mathbf{SkMult}_l & \xrightarrow{U_l} & \mathbf{FSkMult}_l \\ {\scriptstyle T_l}\downarrow & \swarrow {\scriptstyle K_l} & \\ \mathbf{Skew} & & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbf{SkMult}^{cl} & \xrightarrow{U^{cl}} & \mathbf{FSkMult}^{cl} \\ {\scriptstyle T^{cl}}\downarrow & \swarrow {\scriptstyle K^{cl}} & \\ \mathbf{SkewCl}. & & \end{array}$$

There are also equivalences with finitary left representable closed skew multicategories.

Bonus Track

Biclosed Multicategories

In a **biclosed** multicategory we have isomorphisms

$$\lambda_n \colon \mathcal{C}_n(a_1, \ldots, a_n; b) \cong \mathcal{C}_{n-1}(a_1, \ldots, a_{n-1}; l[a_n, b]) \quad \text{and}$$
$$\rho_n \colon \mathcal{C}_n(a_1, \ldots, a_n; b) \cong \mathcal{C}_{n-1}(a_2, \ldots, a_n; r[a_1, b]).$$

$\Rightarrow$ we can define $\circ_i$ using these isomorphisms.

Let $f \colon a_1, a_2 \to b_2$ and $g \colon b_1, b_2, b_3 \to c$, then we want $g \circ_2 f \colon b_1, a_1, a_2, b_3 \to c$

$$\cfrac{\cfrac{g \colon b_1, b_2, b_3 \to c}{\lambda_3 g \colon b_1, b_2 \to l[b_3, c]}}{\rho_2(\lambda_3 g) \colon b_2 \to r[\, b_1, l[b_3, c]\,]}$$

Then, $\rho_2(\lambda_3 g) \circ f \colon a_1, a_2 \to r[\, b_1, l[b_3, c]\,]$ and back using ρ_3^{-1} and λ_4^{-1}.

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$$\mathcal{C}_m(\bar{b}; c) \times \mathcal{C}_n(\bar{a}; b_i)$$

$$\downarrow \Phi$$

$$\mathcal{C}_1(b_i; r[b_{i-1}, \ldots r[b_1, l[b_{i+1}, \ldots l[b_m, c]\ldots]]]) \times \mathcal{C}_n(\bar{a}; b_i)$$

$$\downarrow P$$

$$\mathcal{C}_n(\bar{a}; r[b_{i-1}, \ldots r[b_1, l[b_{i+1}, \ldots l[b_m, c]\ldots]]])$$

$$\downarrow \Psi$$

$$\mathcal{C}_{n+m-1}(b_1, \ldots b_{i-1}, \bar{a}, b_{i+1}, \ldots, b_n; c)$$

where Φ is a composition of λ's and ρ's, P is the profunctor action of $\mathcal{C}_n(-;-)$ and Ψ is a composition of λ^{-1}'s and ρ^{-1}'s.

Proposition

Biclosed multicategories can be defined as biclosed families of profunctors $(\mathcal{C}, \lambda, \rho)$ satisfying some extra equations (corresponding to the associativity equations).

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