Constructing Canonical Calculi

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Plan of Action: The 3 W's

1. Why?

- Backstory
- Motivation
- Aim

2. Who?

- Setting
- Preliminaries

3. How?

- New ideas!
- Main Results

The Backstory

- **Noncommutative Geometry:** $A \in \mathbb{K}\text{-Alg}$ + Extra Structure
  $\rightsquigarrow$ "non comm. space"

- One way is using Differential Calculi, i.e.
  dg-Alg. $\Omega^\cdot$ gen. in degree zero $\Omega^0 = A$
  via the monoid prod. & differentials $d^n : \Omega^n \longrightarrow \Omega^{n+1}$

- $\Omega^1 :=$ 1st ord. diff. calc. $\rightsquigarrow$ non comm. bimod. of sections of cotang. bundle

- Usual Approach: $A \rightsquigarrow \Omega^1_{U,A} :=$ univ. 1st ord. calc. ---> WHO
  $\rightsquigarrow \Omega^1_{U,A}/\sim$ additional structure ---> HOW

A PROBLEM... ?

- $\exists\ (-)_{max} : \mathrm{Calc}' \to \mathrm{Calc}^{\bullet}$ canonical functorial ☺
  ↳ "maximal prolungation"
- "Algebras" $\dashrightarrow \mathrm{Calc}'$ functorial ? ☹
- Aim: Find functor "Algebras" $\to \mathrm{Calc}^{\bullet}$
  ↳ Steps
  - Set a notion of "Algebras"
  - Find "Alg." $\to \mathrm{Calc}'$
  - $\underbrace{\text{"Alg."} \to \mathrm{Calc}' \xrightarrow{(-)_{max}} \mathrm{Calc}^{\bullet}}_{\text{desired functor}}$

THE SETTING

- We work w/ a monoidal additive cat. $\mathcal{V}$

  i.e.
  - $(\mathcal{V}, \otimes, I)$ mon. cat.
  - $\mathcal{V}$ Ab-enriched (Mon-enriched enough for some things)
  - $\otimes$ additive in each variable
  - $\mathcal{V}$ has finite lim. & colim.

- Main case: $\mathcal{V}$ loc. presentable & $\otimes$ accessible in each variable

Memo: Monoids & Bimodules in $\mathcal{V}$

- $\mathrm{Mon}(\mathcal{V})$:
  - (A, m, i) monoid in $\mathcal{V}$ $\rightsquigarrow$ $m: A \otimes A \to A$ $\quad i: I \to A$ assoc. & unital
  - $f: A \to B \in \mathcal{V}$ respecting m & i
- $\mathrm{Mod}(\mathcal{V})$:
  - (A, M) w/ $A \in \mathrm{Mon}(\mathcal{V})$ & M an A-bimod. $\rightsquigarrow$ $l_M: A \otimes M \to M$, $r_M: M \otimes A \to M$ assoc. & unital
  - $(f, g): (A, M) \to (B, N)$ w/ $f: A \to B \in \mathrm{Mon}(\mathcal{V})$ & $g: M \to f^* N \in {}_A\mathrm{Mod}_A$ (i.e. respects actions)

    extension of scalars

    e.g. $A \otimes N \xrightarrow{f \otimes 1_N} B \otimes N \to N$

1^{st} Order Differential Calculi

- $A \in \mathrm{Mon}(\mathcal{V})$, a 1^{st} ord. diff. calc. Ω^1_d consists of (Ω^1_d, l_d, r_d) an A-bimod t.w/ $d: A \to \Omega^1_d$ s.t.

LEIBNIZ RULE (1)

$$\begin{tikzcd} A\otimes A \arrow[r, "m"] \arrow[rr, bend right, "d\cdot 1_A + 1_A\cdot d"'] & A \arrow[r, "d"] & \Omega^1_d \end{tikzcd}$$

" $d(x\cdot y) = d(x)\cdot y + x\cdot d(y)$ "

w/ $d\cdot 1_A := r_d \circ d\otimes 1_A$
& $1_A\cdot d := l_d \circ 1_A \otimes d$

SURJ. CONDITION (2)

$$\begin{tikzcd} A\otimes A \arrow[rr, two heads, "1_A\cdot d"] \arrow[dr, "1_A\otimes d"'] & {} \arrow[d, phantom, "\shortparallel"] & \Omega^1_d \\ & A\otimes\Omega^1_d \arrow[ur, "l_d"'] & \end{tikzcd}$$

is epi " Ω^1_d gen. by dA as left mod "

Prop If (1) : $\underset{\overset{\shortparallel}{(2)}}{1_A\cdot d \text{ epi}} \iff d\cdot 1_A$ epi $\iff 1_A\cdot d\cdot 1_A$ epi

The Universal 1st Order Diff. Calculus

- $$\begin{tikzcd} \Omega^1_{U,A} \arrow[r, tail, "\iota_U := \mathrm{Ker}(m)"] & A \otimes A \arrow[r, "m"] & A \\ A \arrow[u, dashed, "\exists !\, d_U"] \arrow[ur, "i \otimes 1_A - 1_A \otimes i"'] & & \end{tikzcd}$$

$\rightsquigarrow$

$$\begin{tikzcd} \Omega^1_{U,A} \otimes A \arrow[r, "\iota_U \otimes 1_A"] \arrow[d, dashed, "\exists !\, r_U"'] & A \otimes A \otimes A \arrow[r, "m \otimes 1_A"] \arrow[d, "1_A \otimes m"] & A \otimes A \arrow[d, "m"] \\ \Omega^1_{U,A} \arrow[r, tail] & A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

Similarly we define l_U

Prop $(\Omega^1_{U,A}, l_U, r_U) \in {}_A\mathrm{Mod}_A$ & $(\Omega^1_{U,A}, d) \in \mathrm{Calc}^1_A$

Prop Ω^1_d 1st ord. diff. calc. over A $\iff$

$$\begin{tikzcd} \Omega^1_U \arrow[rr, two heads, "\exists (!)"] & & \Omega^1_d \\ & A \arrow[ul, dashed, "d_U"] \arrow[ur, dashed, "d"'] & \end{tikzcd}$$

epi in $\mathcal{V}$

The Square-Zero Extention

- $U: \mathrm{Mod}(\mathcal{V}) \longrightarrow \mathrm{Mon}(\mathcal{V})$ is the "square-zero extension"
  $(A, M) \longmapsto A \oplus M$

$$\begin{tikzcd}
(A\oplus M)\otimes(A\oplus M) \cong (A\otimes A)\oplus(A\otimes M)\oplus(M\otimes A)\oplus(M\otimes M) \arrow[d, "m_A + l_M + r_M + 0"] \arrow[d, bend right, "m_{A\oplus M}"', "=:"] \\
A \oplus M
\end{tikzcd}$$

Prop

$$\begin{tikzcd}
A \arrow[d, mapsto] & \mathrm{Mon}(\mathcal{V}) \arrow[d, bend right, "L"'] \arrow[d, phantom, "\dashv"] \\
(A, \Omega^1_U) & \mathrm{Mod}(\mathcal{V}) \arrow[u, bend right, "U"']
\end{tikzcd}$$

& the unit is

$$(1_A, d_U): A \longrightarrow A \oplus \Omega^1_U = UL(A)$$

The Pullback Construction

- Idea: "Algebra" $\leftrightsquigarrow$ faithful isofibration $(-)_0 : \mathcal{E} \to \mathrm{Mon}(\mathcal{V})$

- Need a notion of "bimodules over $\mathcal{E}$"

$$\begin{tikzcd} \mathrm{Mod}(\mathcal{E}) \arrow[r, "K"] \arrow[d, "U'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Mod}(\mathcal{V}) \arrow[d, "U"] \\ \mathcal{E} \arrow[r, "(-)_0"'] & \mathrm{Mon}(\mathcal{V}) \end{tikzcd}$$

- Examples

$\mathcal{V}$	$\mathcal{E}$	$(A,M) \in \mathrm{Mod}(\mathcal{E})$
braided	$\mathrm{CMon}(\mathcal{V})$	M central[(*)] bimod. on A
$\mathbb{R}$-Vect	C^∞-Ring	M central bimod. on A_0

(*)
$$\begin{tikzcd} A\otimes M \arrow[r, "\sim"] \arrow[dr, bend right, "l_M"'] & M\otimes A \arrow[r, "\sim"] \arrow[d, "r_M"] & A\otimes M \arrow[dl, bend left, "l_M"] \\ & M & \end{tikzcd}$$

Weakly Geometrical Functors

- When does $\mathrm{Mod}(\mathcal{E})$ look like a cat. of bimodules?

  $\rightsquigarrow$ **Def** $(-)_0$ weakly $\mathcal{V}$-geometrical $\rightsquigarrow$ "LIFTING PROPERTY"

- Examples: $C\,\mathrm{Mon}(\mathcal{V}) \hookrightarrow \mathrm{Mon}(\mathcal{V})$, $\mathcal{C}^\infty\text{-Ring} \longrightarrow \mathrm{Mon}(\mathbb{R}\text{-Vect})$

$\forall\ E \in C\mathrm{Mon}(\mathcal{V})$ $\quad \forall\ E \xrightarrow{(f_1,f_2)} A \oplus M \Rightarrow f_1^* M$ is a central bimod

$(A,M) \in \mathrm{Mod}(C\mathrm{Mon}(\mathcal{V}))$ $\quad$ i.e. $(E, f_1^* M) \in \mathrm{Mod}(C\mathrm{Mon}(\mathcal{V}))$

- Now we can consider GENERALISED 1st order $\mathcal{E}$-calc.

Generalised 1st Order $\mathcal{E}$-Diff. Calculus

Def $E \in \mathcal{E}$, a generalised 1^{st} ord diff. calc. over E is $\Omega \in {}_E\mathrm{Mod}_E$

w/ $d : E_0 \to \Omega_0 \in \mathcal{V}$ s.t.

$$\begin{tikzcd}
{}_E\mathrm{Mod}_E \arrow[r, "(-)_0"] \arrow[d, "U'_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & {}_{E_0}\mathrm{Mod}_{E_0} \arrow[d] \\
\mathcal{E} \arrow[r, "(-)_0"'] & \mathrm{Mon}(\mathcal{V})
\end{tikzcd}$$

(STRONG LEIBNIZ) $d : E_0 \to \Omega_0$ satisfy Leibniz

& lifts to $\hat{d} : E \to U'_E(\Omega) \in \mathcal{E}$.

Prop $(-)_0$ weakly geom., if $\exists\, L' \dashv U' : \mathrm{Mod}(\mathcal{E}) \to \mathcal{E}$ then $\forall\, E \in \mathcal{E}$

$L'E = (\Omega, d : E_0 \to \Omega_0)$ is a gen. 1^{st} ord. diff. calc. w/

$\hat{d} := \eta_E$ & $\hat{d}_0 = (1_{E_0}, d) : E_0 \to E_0 \oplus \Omega_0$ its underlying component

And Now To 1st Ord. $\mathcal{E}$-Diff. Calculus

Def $(-)_0$ is $\mathcal{V}$-geometrical if it is weakly geom. & $\forall E \in \mathcal{E}$

$${}_E\mathrm{Mod}_E \longrightarrow \mathcal{V}$$
$$M \longmapsto \text{underlying obj of } M_0$$

has a left adjoint L_E

Def $(-)_0$ geom., a 1st ord. $\mathcal{E}$-diff. calc. over $E \in \mathcal{E}$ is $\Omega \in {}_E\mathrm{Mod}_E$ w/ $d : E_0 \to \Omega_0 \in \mathcal{V}$ s.t.

(1) Strong Leibniz

(2) $d^t : L_E(E_0) \to \Omega$ epi in ${}_E\mathrm{Mod}_E$ $\rightsquigarrow$ SURJECTIVITY: d generates Ω as an E-bimod

Prop $(-)_0$ $\mathcal{V}$-geom. If $\exists L' \dashv U' : \mathrm{Mod}(\mathcal{E}) \to \mathcal{E}$ $\forall E \in \mathcal{E}$

$(L'E, \eta_E : E \to U'L'E)$ is a 1st ord. $\mathcal{E}$-diff. calc.

The Main Result

- Memo: $\mathcal{V}$ loc. pres., $\otimes$ accessible in each variable

$\Rightarrow$ $\mathrm{Mon}(\mathcal{V})$, $\mathrm{Mod}(\mathcal{V})$ loc. pres. & $U : \mathrm{Mod}(\mathcal{V}) \to \mathrm{Mon}(\mathcal{V})$ cont. and accessible

- $\mathcal{V}$ loc. pres., add. mon. cat. w/ $\otimes$ acc. in each variable (*)

If $\mathcal{E}$ loc. pres. $(-)_0$ accessible & cont. $\Rightarrow$ $\mathrm{Mod}(\mathcal{E})$ loc. pres. & $\exists$ $L' \dashv U' : \mathrm{Mod}(\mathcal{E}) \to \mathcal{E}$, $L'_E \dashv U'_E : {}_E\mathrm{Mod}_E \to \mathcal{V}$

Thm (L., Flood, Tendas) $\mathcal{V}$ as above (*), $\mathcal{E}$ loc. pres., $(-)_0$ weakly geom. acc. & cont.

$\Rightarrow$ $(-)_0$ $\mathcal{V}$-geom., $\exists$ $L' \dashv U'$ & $L'E$ 1^{st} ord. $\mathcal{E}$-diff. calc.

EXAMPLES

- $\mathrm{Mon}(\mathcal{V}) = \mathrm{Mon}(\mathcal{V}) \rightsquigarrow$ universal calc.

- $\mathrm{CMon}(\mathcal{V}) \hookrightarrow \mathrm{Mon}(\mathcal{V})$ weakly $\mathcal{V}$-geom. so, if $\mathcal{V}$ loc. pres. $\Rightarrow$ $\mathcal{V}$-geom.

  If $\mathcal{V} = \mathbb{K}\text{-Vect} \Rightarrow \mathcal{L}'$ gives the Kähler Diff.

- $\mathrm{CMon}_m(\mathcal{V}) \hookrightarrow \mathrm{Mon}(\mathcal{V})$ m-comm. monoids $\rightsquigarrow$

$$\begin{tikzcd} A \otimes A \arrow[rr, "\beta^m"] \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

  ↳ weakly $\mathcal{V}$-geom.

- $C^\infty\text{-Ring} \to \mathrm{Mon}(\mathbb{R}\text{-Vect})$ is $(\mathbb{R}\text{-Vect})$-geom.

  ↳ $L_\infty(E) = (\Omega^\infty, d^\infty : E_0 \to \Omega^\infty_0)$ w/ d^∞ a C^∞-derivation

From "Algebras" to $\mathrm{Calc}'(\mathcal{V})$

Memo: $\mathcal{V}$ loc. pres., add. mon. cat. w/ $\otimes$ cocont. in each variable

$\mathcal{E}$ loc. pres., $(-)_0 : \mathcal{E} \to \mathrm{Mon}(\mathcal{V})$ accessible, cocont. & weakly $\mathcal{V}$-geom.

Cor $\exists$ functor $\mathrm{Calc}'_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathrm{Calc}'(\mathcal{V})$

"Proof" $E \longmapsto L'E = (\Omega, d : E_0 \to \Omega_0)$ 1^{st} ord. $\mathcal{E}$-diff. calc.

Missing: "surjectivity over E_0"

$$L \dashv U \quad \frac{(1_{E_0}, d) : E_0 \to E_0 \oplus \Omega_0 = U(E_0, \Omega_0)}{h : \Omega'_{U,E_0} \to \Omega_0} \quad \text{NOT EPI}$$

$\rightsquigarrow$

$$\begin{tikzcd} \Omega'_{U,E_0} \arrow[rr, "h"] \arrow[dr, two heads, "\text{epi}"'] & & \Omega_0 \\ & \Omega'_0 \arrow[ur, tail, "\text{str. mono}"'] & \end{tikzcd}$$

$\Rightarrow \mathrm{Calc}'_{\mathcal{E}}(E) := \Omega'_0$ "$\square$"

SUMMARY

- Notion of "Algebras" $\longleftrightarrow$ $\mathcal{E}$ w/ faithful isofibration $(-)_0 : \mathcal{E} \longrightarrow \mathrm{Mon}(\mathcal{V})$
- If $(-)_0$ weakly $\mathcal{V}$-geom. $\rightsquigarrow$ canonical generalised 1^{st} order $\mathcal{E}$-diff. calc.
- If $(-)_0$ $\mathcal{V}$-geom. $\rightsquigarrow$ canonical 1^{st} order $\mathcal{E}$-diff. calc.
- If $\mathcal{V}$ & $\mathcal{E}$ "nice" & $(-)_0$ weakly $\mathcal{V}$-geom.

$\Rightarrow \mathrm{Calc}^1_{\mathcal{E}} : \mathcal{E} \longrightarrow \mathrm{Calc}^1(\mathcal{V})$ ... and $\mathcal{E} \longrightarrow \mathrm{Calc}^{\bullet}(\mathcal{V})$?

SHORT ANSWER $(-)_{\mathrm{Max}}$ works ☺ LONG ANSWER Paper!

The End

Thanks for Listening

:)

EXTRA

000 EVIDENCE →

DIFFERENTIAL CALCULI

<u>Def</u> a differential calculus $\Omega^\bullet_d$ over $A \in Mon(\mathcal{V})$ is

$(\Omega^\bullet_d = \bigoplus_{m \geq 0} \Omega^m_d, \wedge, d) \in Mon(CoCh(\mathcal{V}))$ s.t.

- $\Omega^0_d = A$, $\wedge^{0,0} = m_A$ & $1_{\Omega^\bullet_d} = 1_A$;

SURJ. CONDITION

- $\forall m$ the composite below is an epi

$$A^{\otimes m+1} \xrightarrow{1_A \otimes d^{\otimes m}} A \otimes \Omega^1_d \otimes (\Omega^1_d)^{\otimes m-1} \xrightarrow{1_d \otimes 1_{(\Omega^1_d)^{\otimes m-1}}} (\Omega^1_d)^{\otimes m} \xrightarrow{\wedge} \Omega^m_d$$

- $Calc^\bullet(\mathcal{V}) \hookrightarrow Mon(CoCh(\mathcal{V}))$ full subcat. of diff. calculi

- $\forall M^\bullet \in Mon(CoCh(\mathcal{V})) \Rightarrow M^0$ monoid & $\forall m \geq 1$ $M^m \in {}_{M^0}Mod_{M^0}$

$\hookrightarrow$ $Calc^\bullet(\mathcal{V}) \longrightarrow Calc^1(\mathcal{V})$

$$\Omega^\bullet_d \longmapsto (\Omega^0_d, \Omega^1_d)$$

THE MAXIMAL PROLONGATION

- Aim: $\forall\, (A, \Omega^1_d) \in \mathrm{Calc}^1_c(\gamma) \rightsquigarrow \Omega^\bullet_{d,max} \in \mathrm{Calc}^\bullet_c(\gamma)$ universal
- First: $\Omega^1_\upsilon \rightsquigarrow \Omega^\bullet_{\upsilon,max}$ w/ $\Omega^m_{\upsilon,max} := (\Omega^1_\upsilon)^{\otimes m}$ ...
- $(A, \Omega^1_d) \in \mathrm{Calc}^1_c(\gamma)$ the maximal prolongation $\Omega^\bullet_{d,max}$ is def. as:
  - $\Omega^0_{d,max} := A$, $\eta := i_A$, $\wedge^{0,0}_d := m_A$
  - $\Omega^1_{d,max} := \Omega^1_d$, $\wedge^{0,1}_d := l_d$ & $\wedge^{1,0}_d := r_d$
  - $m \geq 2$ Ω^m_d def. (recur.) as the colim. below

$$\begin{tikzcd}
\bigoplus_{i+j=m}^{i,j\geq 1} \Omega^i_\upsilon \otimes \Omega^j_\upsilon \arrow[r, "\wedge_\upsilon"] \arrow[d, "f^i \otimes f^j"'] & \Omega^m_\upsilon \arrow[d, dashed, "f^m"] & \Omega^{m-1}_\upsilon \arrow[l, "d^{m-1}_\upsilon"'] \arrow[d, "f^{m-1}"] \\
\bigoplus_{i+j=m}^{i,j\geq 1} \Omega^i_d \otimes \Omega^j_d \arrow[r, dashed, "\wedge_d"] & \Omega^m_d & \Omega^{m-1}_d \arrow[l, dashed, "d^{m-1}"']
\end{tikzcd}$$

de Rham Functors

- $(-)_{max} : \mathrm{Calc}^1(\mathcal{V}) \longrightarrow \mathrm{Calc}^\bullet(\mathcal{V})$

Prop $\pi : \mathrm{Calc}^\bullet(\mathcal{V}) \longrightarrow \mathrm{Calc}^1(\mathcal{V})$

$\Omega^\bullet_d \longmapsto (\Omega^0, \Omega^1)$

$\exists \; (-)_{max}$

$$\begin{tikzcd}
\mathrm{Calc}^\bullet(\mathcal{V}) \arrow[d, "\pi"] & A, \Omega^1_d, 0, 0 \ldots \\
\mathrm{Calc}^1(\mathcal{V}) \arrow[u, bend left=50, "(-)_{max}"] \arrow[u, bend right=50] \arrow[u, phantom, shift left=3ex, "\dashv"] \arrow[u, phantom, shift right=3ex, "\dashv"] & (A, \Omega^1_d) \arrow[u, mapsto]
\end{tikzcd}$$

- $\forall\, \mathcal{E}$ "nice", its de Rham functor $dR_\mathcal{E}$ is $\mathcal{E} \xrightarrow{\mathrm{Calc}^1_\mathcal{E}} \mathrm{Calc}^1(\mathcal{V}) \xrightarrow{(-)_{max}} \mathrm{Calc}^\bullet(\mathcal{V})$

Prop $\forall\, e_1, e_2$ "nice",

$$\begin{tikzcd}
\mathcal{E}_1 \arrow[d, "k"'] \arrow[dr, "e_1"] & \\
\mathcal{E}_2 \arrow[ur, phantom, "\cong"] \arrow[r, "e_2"'] & \mathrm{Mon}(\mathcal{V})
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
\mathcal{E}_1 \arrow[d, "k"'] \arrow[dr, "dR_{\mathcal{E}_1}"] & \\
\mathcal{E}_2 \arrow[ur, phantom, "\Downarrow"] \arrow[r, "dR_{\mathcal{E}_2}"'] & \mathrm{Calc}^\bullet(\mathcal{V})
\end{tikzcd}$$

Fixing an Almost Example: Banach Spaces

- $\mathcal{V} = \mathrm{Ban}_b$ = Banach spaces & bounded maps
  ↳ additive, fin. co/lim., $\otimes$ preserves co/ker's

- $\mathrm{Ban}_1 \longrightarrow \mathrm{Ban}_b$: same objects & linear contractions
  ↳ co/complete, loc. pres., symm. mon. closed ... NOT additive

- $$\begin{tikzcd}
\mathrm{Mod}(\mathrm{Ban}_1) \arrow[r, tail] \arrow[d, "U_1"'] \arrow[d, phantom, shift left=4ex, "\dashv"] & \mathrm{Mod}(\mathrm{Ban}_b) \arrow[d, "U_b"] \arrow[d, phantom, shift left=4ex, "\vdash"] \\
A \in \mathrm{Mon}(\mathrm{Ban}_1) \arrow[r, tail, "(-)_0"'] & \mathrm{Mon}(\mathrm{Ban}_b) \arrow[u, dashed, bend right, "L_b"']
\end{tikzcd}$$

  $\mathrm{Mon}(\mathrm{Ban}_1)$ = Banach Algebras

  - (Probably) U_1 does not preserve products
  - $L_b(A_0) \in \mathrm{Mod}(\mathrm{Ban}_1)$
  - $\mathrm{Mod}(\mathrm{Ban}_1)\big((A, \Omega^1_A), (B,N)\big) \cong \mathrm{Mod}(\mathrm{Ban}_b)\big(A, U_1(B,N)\big)_{(1, \|d\|)}$