


Sketches and Classifying Logoi

CT2024, Santiago de Compostela

June 26, 2024

Gabriele Lobbia¹

Università di Bologna

 globbia.github.io/seminars/

Preprint: arXiv:2403.09264



¹joint work with Ivan Di Liberti




What's the Story?

- Logical operations ^{described with}  limits & colimits




What's the Story?

- Logical operations $\overset{\text{described with}}{\longleftrightarrow}$ limits & colimits
- Geometric Logic, i.e. arbitrary \vee & fin. \wedge w/ inf. distr. rule
 \rightsquigarrow Topoi (In this talk: Topos = Grothendieck Topos !)

What's the Story?

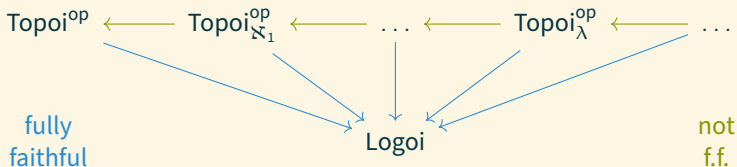
- Logical operations ^{described with}  limits & colimits
- Geometric Logic, i.e. arbitrary \vee & fin. \wedge w/ inf. distr. rule
 Topoi (In this talk: Topos = Grothendieck Topos !)
- λ -ary Geometric Logic, i.e. arbitrary \vee & ($< \lambda$)-ary \wedge (w/ distr. rule)
 Topoi $_{\lambda}$ (E.g. Geometric = \aleph_0 -ary Geometric)

What's the Story?

- Logical operations ^{described with}  limits & colimits
- Geometric Logic, i.e. arbitrary \vee & fin. \wedge w/ inf. distr. rule
 Topoi (In this talk: Topos = Grothendieck Topos !)
- λ -ary Geometric Logic, i.e. arbitrary \vee & ($< \lambda$)-ary \wedge (w/ distr. rule)
 Topoi $_{\lambda}$ (E.g. Geometric = \aleph_0 -ary Geometric)
- **Aim:** Notion of Logos capturing all infinitary logics.

What's the Story?

- Logical operations $\xleftrightarrow{\text{described with}}$ limits & colimits
- Geometric Logic, i.e. arbitrary \vee & fin. \wedge w/ inf. distr. rule
 \rightsquigarrow Topoi (In this talk: Topos = Grothendieck Topos !)
- λ -ary Geometric Logic, i.e. arbitrary \vee & $(< \lambda)$ -ary \wedge (w/ distr. rule)
 \rightsquigarrow Topoi_λ (E.g. Geometric = \aleph_0 -ary Geometric)
- **Aim:** Notion of Logos capturing all infinitary logics.



The Aim: Presentations & Classifying Objects

Logic Fragment	Presentation	Classifying Object²
Geometric	Site	Topos
Infinitary	?	?

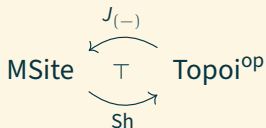
²i.e. the *syntax independent* presentation

The Aim: Presentations & Classifying Objects

Logic Fragment	Presentation	Classifying Object ²
Geometric	Site	Topos
Infinitary	?	?

Theorem (Diaconescu for Topoi)

Topoi is (bi)reflective^a in Morita small sites.



^ai.e. biadjunction w/ counit a component-wise equivalence

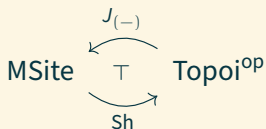
²i.e. the *syntax independent* presentation

The Aim: Presentations & Classifying Objects

Logic Fragment	Presentation	Classifying Object ²
Geometric	Site	Topos
Infinitary	?	?

Theorem (Diaconescu for Topoi)

Topoi is (bi)reflective^a in Morita small sites.



^ai.e. biadjunction w/ counit a component-wise equivalence

- **Aim:** Fill in the table.

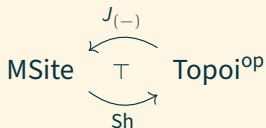
²i.e. the *syntax independent* presentation

The Aim: Presentations & Classifying Objects

Logic Fragment	Presentation	Classifying Object ²
Geometric	Site	Topos
Infinitary	?	?

Theorem (Diaconescu for Topoi)

Topoi is (bi)reflective^a in Morita small sites.



^ai.e. biadjunction w/ counit a component-wise equivalence

- **Aim:** Fill in the table.
- **How:** Diaconescu for *sketches*.

²i.e. the *syntax independent* presentation

The Main Character: Sketches

- A **sketch** $\mathcal{S} = (\mathbb{S}, L_{\mathcal{S}}, C_{\mathcal{S}})$ consists of a locally small cat. \mathbb{S} w/

The Main Character: Sketches

- A **sketch** $\mathcal{S} = (\mathbb{S}, L_{\mathcal{S}}, C_{\mathcal{S}})$ consists of a locally small cat. \mathbb{S} w/
 - a class of (ess. small) cones $L_{\mathcal{S}}$ \longleftrightarrow *limit part*
 - a class of (ess. small) cocones $C_{\mathcal{S}}$ \longleftrightarrow *colimit part*

The Main Character: Sketches

- A **sketch** $\mathcal{S} = (\mathbb{S}, L_{\mathcal{S}}, C_{\mathcal{S}})$ consists of a locally small cat. \mathbb{S} w/
 - a class of (ess. small) cones $L_{\mathcal{S}} \rightsquigarrow$ *limit part*
 - a class of (ess. small) cocones $C_{\mathcal{S}} \rightsquigarrow$ *colimit part*
- A **sketch morphism** $F: \mathcal{S} \rightarrow \mathcal{T}$ is a functor $F: \mathbb{S} \rightarrow \mathbb{T}$ s.t.
$$\forall \delta \in L_{\mathcal{S}}/C_{\mathcal{S}} \quad \exists \pi \in L_{\mathcal{T}}/C_{\mathcal{T}} \quad \text{s.t. } F\delta \cong \pi$$

The Main Character: Sketches

- A **sketch** $\mathcal{S} = (\mathbb{S}, L_{\mathcal{S}}, C_{\mathcal{S}})$ consists of a locally small cat. \mathbb{S} w/
 - a class of (ess. small) cones $L_{\mathcal{S}} \rightsquigarrow$ *limit part*
 - a class of (ess. small) cocones $C_{\mathcal{S}} \rightsquigarrow$ *colimit part*
- A **sketch morphism** $F: \mathcal{S} \rightarrow \mathcal{T}$ is a functor $F: \mathbb{S} \rightarrow \mathbb{T}$ s.t.
$$\forall \delta \in L_{\mathcal{S}}/C_{\mathcal{S}} \quad \exists \pi \in L_{\mathcal{T}}/C_{\mathcal{T}} \quad \text{s.t. } F\delta \cong \pi$$
- $\text{Skt} = 2\text{-category of sketches (2-cells = nat. transf.)}$

The Main Character: Sketches

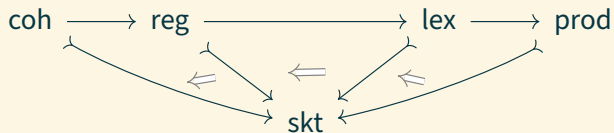
- A **sketch** $\mathcal{S} = (\mathbb{S}, L_{\mathcal{S}}, C_{\mathcal{S}})$ consists of a locally small cat. \mathbb{S} w/
 - a class of (ess. small) cones $L_{\mathcal{S}} \rightsquigarrow$ *limit part*
 - a class of (ess. small) cocones $C_{\mathcal{S}} \rightsquigarrow$ *colimit part*
- A **sketch morphism** $F: \mathcal{S} \rightarrow \mathcal{T}$ is a functor $F: \mathbb{S} \rightarrow \mathbb{T}$ s.t.
$$\forall \delta \in L_{\mathcal{S}}/C_{\mathcal{S}} \quad \exists \pi \in L_{\mathcal{T}}/C_{\mathcal{T}} \quad \text{s.t. } F\delta \cong \pi$$
- $\text{Skt} =$ 2-category of sketches (2-cells = nat. transf.)

Example

\mathbb{C} cat. w/ finite products $\mapsto (\mathbb{C}, \text{fin. prod. cones}, \emptyset) \in \text{Skt}$

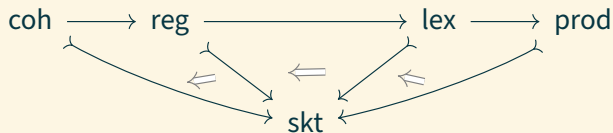
More Examples!

- **First order doctrines**



More Examples! (...and a sanity check)

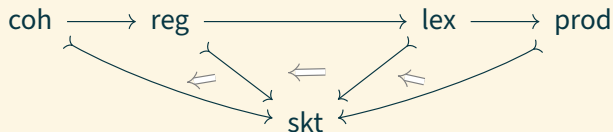
- **First order doctrines**



- **Sanity Check:** \mathbb{E} $\text{topos} \mapsto (\mathbb{E}, \text{fin. lim, all colim}) \in \text{Skt}$

More Examples! (...and a sanity check)

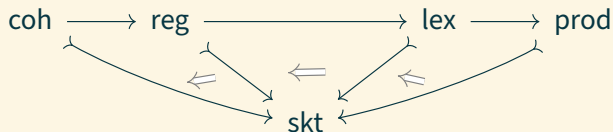
- **First order doctrines**



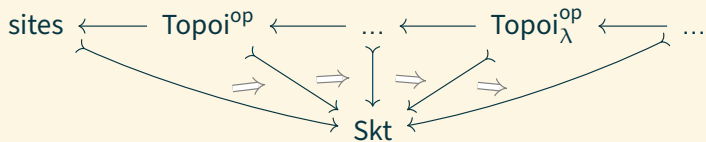
- **Sanity Check:** \mathbb{E} λ -ary topoi $\mapsto (\mathbb{E}, \lambda$ -ary lim, all colim) \in Skt

More Examples! (...and a sanity check)

- **First order doctrines**



- **Sanity Check:** $\mathbb{E} \lambda\text{-ary topoi} \mapsto (\mathbb{E}, \lambda\text{-ary lim, all colim}) \in \text{Skt}$



Interlude: Parametric Morita

- A **model** of a sketch \mathcal{S} in a complete & cocomplete cat \mathbb{C} is a functor sending each **cone/cocone** in \mathcal{S} to **lim/colim** in \mathbb{C} .

³**Notation:** we call these **test** sketches.

Interlude: Parametric Morita

- A **model** of a sketch \mathcal{S} in a complete & cocomplete cat \mathbb{C} is a functor sending each **cone/cocone** in \mathcal{S} to **lim/colim** in \mathbb{C} .

$$\iff M: \mathcal{S} \rightarrow \mathcal{C} \in \text{Skt} \quad \text{w/ } \mathcal{C} = (\mathbb{C}, \text{all lim, all colim})^3$$

³**Notation:** we call these **test** sketches.

Interlude: Parametric Morita

- A **model** of a sketch \mathcal{S} in a complete & cocomplete cat \mathbb{C} is a functor sending each **cone/cocone** in \mathcal{S} to **lim/colim** in \mathbb{C} .

$$\iff M: \mathcal{S} \rightarrow \mathbb{C} \in \text{Skt} \quad \text{w/ } \mathbb{C} = (\mathbb{C}, \text{all lim, all colim})^3$$

- **Idea:** For $\mathbb{M} \subseteq \text{Skt}$, $F: \mathcal{S} \rightarrow \mathcal{T}$ is a **\mathbb{M} -Morita equivalence** iff
 $\forall \mathcal{M} \in \mathbb{M} \quad - \circ F: \text{Skt}(\mathcal{S}, \mathcal{M}) \simeq \text{Skt}(\mathcal{T}, \mathcal{M})$.

E.g. $\mathbb{M} = \text{test sketches}$ \iff having the same *models*

³**Notation:** we call these **test** sketches.

Interlude: Parametric Morita

- A **model** of a sketch \mathcal{S} in a complete & cocomplete cat \mathbb{C} is a functor sending each **cone/cocone** in \mathcal{S} to **lim/colim** in \mathbb{C} .

$$\iff M: \mathcal{S} \rightarrow \mathbb{C} \in \text{Skt} \quad \text{w/ } \mathbb{C} = (\mathbb{C}, \text{all lim, all colim})^3$$

- **Idea:** For $\mathbb{M} \subseteq \text{Skt}$, $F: \mathcal{S} \rightarrow \mathcal{T}$ is a **\mathbb{M} -Morita equivalence** iff
 $\forall \mathcal{M} \in \mathbb{M} \quad - \circ F: \text{Skt}(\mathcal{S}, \mathcal{M}) \simeq \text{Skt}(\mathcal{T}, \mathcal{M})$.

E.g. $\mathbb{M} = \text{test sketches} \iff$ having the same *models*

- Why is this useful?

³**Notation:** we call these **test** sketches.

Interlude: Parametric Morita

- A **model** of a sketch \mathcal{S} in a complete & cocomplete cat \mathbb{C} is a functor sending each **cone/cocone** in \mathcal{S} to **lim/colim** in \mathbb{C} .

$$\iff M: \mathcal{S} \rightarrow \mathbb{C} \in \text{Skt} \quad \text{w/ } \mathbb{C} = (\mathbb{C}, \text{all lim, all colim})^3$$

- **Idea:** For $\mathbb{M} \subseteq \text{Skt}$, $F: \mathcal{S} \rightarrow \mathcal{T}$ is a **\mathbb{M} -Morita equivalence** iff
 $\forall \mathcal{M} \in \mathbb{M} \quad - \circ F: \text{Skt}(\mathcal{S}, \mathcal{M}) \simeq \text{Skt}(\mathcal{T}, \mathcal{M})$.

E.g. $\mathbb{M} = \text{test sketches}$ \iff having the same *models*

- Why is this useful? No spoilers!

³**Notation:** we call these **test** sketches.

Left Sketch Classifier

- $\mathcal{L} \in \text{Skt}$ is **left** iff \mathbb{L} cocomplete and $C_{\mathcal{L}} = \text{all colim diagrams}$.

Left Sketch Classifier

- $\mathcal{L} \in \text{Skt}$ is **left** iff \mathbb{L} cocomplete and $C_{\mathcal{L}} =$ all colim diagrams.

- $\mathcal{S} \in \text{skt}$,

$$\mathcal{S} \xrightarrow{\gamma} \text{Psh}(\mathcal{S})$$

Left Sketch Classifier

- $\mathcal{L} \in \text{Skt}$ is **left** iff \mathbb{L} cocomplete and $C_{\mathcal{L}} = \text{all colim diagrams}$.

- $\mathcal{S} \in \text{skt}$,
$$\mathbb{S} \xrightarrow{Y} \text{Psh}(\mathbb{S})$$
$$L \left(\begin{array}{c} \downarrow \\ \uparrow \end{array} \right)$$
$$\widehat{\mathbb{S}}$$

- $\widehat{\mathbb{S}} := \mathcal{H}^{\perp} = \text{Psh}(\mathbb{S})[\mathcal{E}_{\mathcal{H}}^{-1}]$ w/

$$\mathcal{H} := \{\rho_{\delta} : \text{colim} YD \Rightarrow Ys \mid \delta : D \Rightarrow \Delta s \in C_{\mathcal{S}}\}$$

Left Sketch Classifier

- $\mathcal{L} \in \text{Skt}$ is **left** iff \mathbb{L} cocomplete and $C_{\mathcal{L}} = \text{all colim diagrams}$.

- $\mathcal{S} \in \text{skt}$,

$$\begin{array}{ccc}
 \mathcal{S} & \xrightarrow{Y} & \text{Psh}(\mathcal{S}) \\
 & \searrow J_{\mathcal{S}} & \downarrow L \left(\begin{array}{c} \dashv \\ \dashv \end{array} \right) \\
 & & \widehat{\mathcal{S}}
 \end{array}$$

- $\widehat{\mathcal{S}} := \mathcal{H}^{\perp} = \text{Psh}(\mathcal{S})[\mathcal{E}_{\mathcal{H}}^{-1}]$ w/

$$\mathcal{H} := \{\rho_{\delta} : \text{colim} YD \Rightarrow Ys \mid \delta : D \Rightarrow \Delta s \in C_{\mathcal{S}}\}$$

- $\widehat{\mathcal{S}} := (\widehat{\mathcal{S}}, L_{J_{\mathcal{S}}}^m, \text{all colim diagrams})$ w/

$L_{J_{\mathcal{S}}}^m := \text{min structure making } J_{\mathcal{S}} \text{ a limit-sketch morphism}$

Second Sanity Check: $\widehat{(-)}$ vs Topoi

- (\mathbb{C}, J) site \rightsquigarrow $\text{Sh}(\mathbb{C}) =$ **Classifying Topos of (\mathbb{C}, J)**

$$\text{sites} \xrightarrow{\text{Sh}} \text{Topoi}^{\text{op}}$$

Second Sanity Check: $\widehat{(-)}$ vs Topoi

- (\mathbb{C}, J) site \rightsquigarrow $\text{Sh}(\mathbb{C}) =$ **Classifying Topos of (\mathbb{C}, J)**
- Does $\widehat{(-)}$ restrict to $\text{Sh}(-)$ for sites?

$$\begin{array}{ccc} \text{sites} & \xrightarrow{\text{Sh}} & \text{Topoi}^{\text{op}} \\ \vdots & ? & \vdots \\ \text{skt} & \xrightarrow{\widehat{(-)}} & \text{LSkt} \end{array}$$

Second Sanity Check: $\widehat{(-)}$ vs Topoi

- (\mathbb{C}, J) site \rightsquigarrow $\text{Sh}(\mathbb{C}) =$ **Classifying Topos of (\mathbb{C}, J)**
- Does $\widehat{(-)}$ restrict to $\text{Sh}(-)$ for sites?

$$\begin{array}{ccc} \text{sites} & \xrightarrow{\text{Sh}} & \text{Topoi}^{\text{op}} \\ \downarrow & \Rightarrow & \downarrow \\ \text{skt} & \xrightarrow{\widehat{(-)}} & \text{LSkt} \end{array}$$

Second Sanity Check: $\widehat{(-)}$ vs Topoi

- (\mathbb{C}, J) site \rightsquigarrow $\text{Sh}(\mathbb{C}) =$ **Classifying Topos of (\mathbb{C}, J)**
- Does $\widehat{(-)}$ restrict to $\text{Sh}(-)$ for sites?

$$\begin{array}{ccc} \text{sites} & \xrightarrow{\text{Sh}} & \text{Topoi}^{\text{op}} \\ \downarrow & \Rightarrow & \downarrow \\ \text{skt} & \xrightarrow{\widehat{(-)}} & \text{LSkt} \end{array}$$

- Why is the 2-cell grey?

Second Sanity Check: $\widehat{(-)}$ vs Topoi

- (\mathbb{C}, J) site \rightsquigarrow $\text{Sh}(\mathbb{C}) =$ **Classifying Topos of (\mathbb{C}, J)**
- Does $\widehat{(-)}$ restrict to $\text{Sh}(-)$ for sites?

$$\begin{array}{ccc} \text{sites} & \xrightarrow{\text{Sh}} & \text{Topoi}^{\text{op}} \\ \downarrow & \Rightarrow & \downarrow \\ \text{skt} & \xrightarrow{\widehat{(-)}} & \text{LSkt} \end{array}$$

- Why is the 2-cell grey?
 1. Underlying category is the same!

Second Sanity Check: $\widehat{(-)}$ vs Topoi

- (\mathbb{C}, J) site \rightsquigarrow $\text{Sh}(\mathbb{C}) =$ **Classifying Topos of (\mathbb{C}, J)**
- Does $\widehat{(-)}$ restrict to $\text{Sh}(-)$ for sites?

$$\begin{array}{ccc} \text{sites} & \xrightarrow{\text{Sh}} & \text{Topoi}^{\text{op}} \\ \downarrow & \Rightarrow & \downarrow \\ \text{skt} & \xrightarrow{\widehat{(-)}} & \text{LSkt} \end{array}$$

- Why is the 2-cell grey?
 1. Underlying category is the same!
 2. No topos can see the difference (i.e. same models in any topos).
 \rightsquigarrow component-wise is a **Topoi-Morita equivalence**

Small Inside: Diaconescu for Left Sketches

- \mathcal{S} and $\widehat{\mathcal{S}}$ have the same models (i.e. $J_{\mathcal{S}}$ is a Test-Morita equivalence)!

Small Inside: Diaconescu for Left Sketches

- \mathcal{S} and $\widehat{\mathcal{S}}$ have the same models (i.e. $J_{\mathcal{S}}$ is a Test-Morita equivalence)!
- Moreover, $J_{\mathcal{S}}: \mathcal{S} \rightarrow \widehat{\mathcal{S}}$ is a **Left-Morita equivalence**, i.e.

$$\forall \mathcal{L} \in \text{LSkt} \quad - \circ J_{\mathcal{S}}: \text{Skt}(\mathcal{S}, \mathcal{L}) \simeq \text{Skt}(\widehat{\mathcal{S}}, \mathcal{L}) = \text{LSkt}(\widehat{\mathcal{S}}, \mathcal{L})$$

Small Inside: Diaconescu for Left Sketches

- \mathcal{S} and $\widehat{\mathcal{S}}$ have the same models (i.e. $J_{\mathcal{S}}$ is a Test-Morita equivalence)!
- Moreover, $J_{\mathcal{S}}: \mathcal{S} \rightarrow \widehat{\mathcal{S}}$ is a **Left-Morita equivalence**, i.e.

$$\forall \mathcal{L} \in \text{LSkt} \quad - \circ J_{\mathcal{S}}: \text{Skt}(\mathcal{S}, \mathcal{L}) \simeq \text{Skt}(\widehat{\mathcal{S}}, \mathcal{L}) = \text{LSkt}(\widehat{\mathcal{S}}, \mathcal{L})$$

- $\text{Skt}^{\text{M}} =$ sketches (Left) Morita equivalent to a small sketch.

Small Inside: Diaconescu for Left Sketches

- \mathcal{S} and $\widehat{\mathcal{S}}$ have the same models (i.e. $J_{\mathcal{S}}$ is a Test-Morita equivalence)!
- Moreover, $J_{\mathcal{S}}: \mathcal{S} \rightarrow \widehat{\mathcal{S}}$ is a **Left-Morita equivalence**, i.e.

$$\forall \mathcal{L} \in \text{LSkt} \quad - \circ J_{\mathcal{S}}: \text{Skt}(\mathcal{S}, \mathcal{L}) \simeq \text{Skt}(\widehat{\mathcal{S}}, \mathcal{L}) = \text{LSkt}(\widehat{\mathcal{S}}, \mathcal{L})$$

- $\text{Skt}^{\text{M}} =$ sketches (Left) Morita equivalent to a small sketch.

Theorem (Di Liberti & L.)

LSkt^{M} is (bi)reflective in Skt^{M} .

$$\begin{array}{ccc} & U^{\mathcal{L}} & \\ & \curvearrowleft & \\ \text{Skt}^{\text{M}} & \top & \text{LSkt}^{\text{M}} \\ & \curvearrowright & \\ & \widehat{(-)} & \end{array}$$

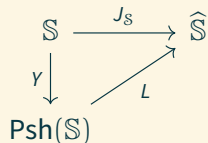
Where the Logoi at?

- $\mathcal{S} \in \text{Skt}$ is (left) rounded if $J_{\mathcal{S}}$ sends cones in $L_{\mathcal{S}}$ to lim diagrams.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{J_{\mathcal{S}}} & \widehat{\mathcal{S}} \\ \gamma \downarrow & \nearrow L & \\ \text{Psh}(\mathcal{S}) & & \end{array}$$

Where the Logoi at?

- $\mathcal{S} \in \text{Skt}$ is (left) rounded if $J_{\mathcal{S}}$ sends cones in $L_{\mathcal{S}}$ to lim diagrams.



E.g. prod, lex,
reg, coh, Topoi

Where the Logoi at?

- $\mathcal{S} \in \text{Skt}$ is (left) rounded if $J_{\mathcal{S}}$ sends cones in $L_{\mathcal{S}}$ to lim diagrams.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{J_{\mathcal{S}}} & \widehat{\mathcal{S}} \\ \gamma \downarrow & \nearrow L & \\ \text{Psh}(\mathcal{S}) & & \end{array}$$

E.g. prod, lex,
reg, coh, Topoi

Proposition

- \mathcal{S} rounded $\Leftrightarrow \widehat{\mathcal{S}}$ is normal (i.e. $L_{\mathcal{S}} \subseteq \text{lim diag.}$ & $C_{\mathcal{S}} \subseteq \text{colim diag.}$)

Where the Logoi at?

- $\mathcal{S} \in \text{Skt}$ is (left) rounded if $J_{\mathcal{S}}$ sends cones in $L_{\mathcal{S}}$ to lim diagrams.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{J_{\mathcal{S}}} & \widehat{\mathcal{S}} \\ \gamma \downarrow & \nearrow L & \\ \text{Psh}(\mathcal{S}) & & \end{array}$$

E.g. prod, lex,
reg, coh, Topoi

Proposition

- \mathcal{S} rounded $\Leftrightarrow \widehat{\mathcal{S}}$ is normal (i.e. $L_{\mathcal{S}} \subseteq \text{lim diag.}$ & $C_{\mathcal{S}} \subseteq \text{colim diag.}$)
- \mathcal{S} rounded $\Rightarrow \widehat{\mathcal{S}}$ rounded

Where the Logoi at?

- $\mathcal{S} \in \text{Skt}$ is (left) rounded if $J_{\mathcal{S}}$ sends cones in $L_{\mathcal{S}}$ to lim diagrams.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{J_{\mathcal{S}}} & \widehat{\mathcal{S}} \\ \gamma \downarrow & \nearrow L & \\ \text{Psh}(\mathcal{S}) & & \end{array}$$

E.g. prod, lex,
reg, coh, Topoi

Proposition

- \mathcal{S} rounded $\Leftrightarrow \widehat{\mathcal{S}}$ is normal (i.e. $L_{\mathcal{S}} \subseteq \text{lim diag.}$ & $C_{\mathcal{S}} \subseteq \text{colim diag.}$)
 - \mathcal{S} rounded $\Rightarrow \widehat{\mathcal{S}}$ rounded
- A logoi \mathcal{S} is a rounded & left sketch.

Where the Logoi at?

- $\mathcal{S} \in \text{Skt}$ is (left) rounded if $J_{\mathcal{S}}$ sends cones in $L_{\mathcal{S}}$ to lim diagrams.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{J_{\mathcal{S}}} & \widehat{\mathcal{S}} \\ \gamma \downarrow & \nearrow L & \\ \text{Psh}(\mathcal{S}) & & \end{array}$$

E.g. prod, lex,
reg, coh, Topoi

Proposition

- \mathcal{S} rounded $\Leftrightarrow \widehat{\mathcal{S}}$ is normal (i.e. $L_{\mathcal{S}} \subseteq \text{lim diag.}$ & $C_{\mathcal{S}} \subseteq \text{colim diag.}$)
 - \mathcal{S} rounded $\Rightarrow \widehat{\mathcal{S}}$ rounded
-
- A logos \mathcal{S} is a rounded & left sketch. **E.g.** Topoi

Where the Logoi at?

- $\mathcal{S} \in \text{Skt}$ is (left) rounded if $J_{\mathcal{S}}$ sends cones in $L_{\mathcal{S}}$ to lim diagrams.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{J_{\mathcal{S}}} & \widehat{\mathcal{S}} \\ \gamma \downarrow & \nearrow L & \\ \text{Psh}(\mathcal{S}) & & \end{array}$$

E.g. prod, lex,
reg, coh, Topoi

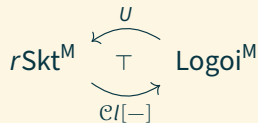
Proposition

- \mathcal{S} rounded $\Leftrightarrow \widehat{\mathcal{S}}$ is normal (i.e. $L_{\mathcal{S}} \subseteq \text{lim diag.}$ & $C_{\mathcal{S}} \subseteq \text{colim diag.}$)
 - \mathcal{S} rounded $\Rightarrow \widehat{\mathcal{S}}$ rounded
-
- A logoi \mathcal{S} is a rounded & left sketch. **E.g.** Topoi, λ -ary Topoi

Classifying Logoi

Theorem (Di Liberti & L.)

Logoi^M is (bi)reflective in $r\text{Skt}^M$.



Classifying Logoi

Theorem (Di Liberti & L.)

Logoi^M is (bi)reflective in $r\text{Skt}^M$.

$$r\text{Skt}^M \begin{array}{c} \xleftarrow{U} \\ \top \\ \xrightarrow{\text{el}[-]} \end{array} \text{Logoi}^M$$

$$\begin{array}{ccc} r\text{Skt}^M & \hookrightarrow & \text{Skt}^M \\ \left. \begin{array}{c} \text{el}[-] \\ \text{U} \end{array} \right\} & & \left. \begin{array}{c} \widehat{(-)} \\ \text{U} \end{array} \right\} \\ \text{Logoi}^M & \hookrightarrow & \text{LSkt}^M \end{array}$$

Classifying Logoi

Theorem (Di Liberti & L.)

Logoi^M is (bi)reflective in $r\text{Skt}^M$.

$$r\text{Skt}^M \begin{array}{c} \xleftarrow{U} \\ \top \\ \xrightarrow{\text{el}[-]} \end{array} \text{Logoi}^M$$

$$\begin{array}{ccccc}
 \text{MSite} & \longrightarrow & r\text{Skt}^M & \hookrightarrow & \text{Skt}^M \\
 \left. \begin{array}{c} \uparrow \text{---} \\ \downarrow \text{---} \end{array} \right\} \text{sh} & \Rightarrow & \left. \begin{array}{c} \uparrow \text{---} \\ \downarrow \text{---} \end{array} \right\} \text{el}[-] & & \left. \begin{array}{c} \uparrow \text{---} \\ \downarrow \text{---} \end{array} \right\} \widehat{(-)} \\
 \text{Topoi}^{\text{op}} & \longrightarrow & \text{Logoi}^M & \hookrightarrow & \text{LSkt}^M
 \end{array}$$

Logic Fragment	Presentation	Classifying Object
Geometric	Site	Topos
Infinitary	Rounded Sketch	Logos

The End

Thanks for listening!