Sketches and Classifying Logoi

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[1]joint work with Ivan Di Liberti

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$$\begin{tikzcd}
\text{Topoi}^{op} \arrow[drr] & \text{Topoi}^{op}_{\aleph_1} \arrow[l] \arrow[dr] & \dots \arrow[l] \arrow[d] & \text{Topoi}^{op}_{\lambda} \arrow[l] \arrow[dl] & \dots \arrow[l] \arrow[dll] \\
& & \text{Logoi} & &
\end{tikzcd}$$

fully faithful

not f.f.

The Aim: Presentations & Classifying Objects

Logic Fragment	**Presentation**	**Classifying Object**[2]
Geometric	Site	Topos
Infinitary	?	?

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$$\begin{tikzcd} \mathsf{MSite} \arrow[r, bend right, "\mathsf{Sh}"'] \arrow[r, phantom, "\top"] & \mathsf{Topoi}^{\mathsf{op}} \arrow[l, bend right, "J_{(-)}"'] \end{tikzcd}$$

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- **How:** Diaconescu for *sketches*.

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The Main Character: Sketches

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$$\forall\, \delta \in \mathsf{L}_{\mathcal{S}}/\mathsf{C}_{\mathcal{S}} \quad \exists\, \pi \in \mathsf{L}_{\mathcal{T}}/\mathsf{C}_{\mathcal{T}} \quad \text{s.t. } F\delta \cong \pi$$

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Example

$\mathbb{C}$ cat. w/ finite products $\mapsto (\mathbb{C}$, fin. prod. cones, $\emptyset) \in$ Skt

- **First order doctrines**

$$
\begin{tikzcd}
\mathrm{coh} \arrow[r, tail] \arrow[drr, tail, bend right] & \mathrm{reg} \arrow[rr, tail] \arrow[dr, tail] \arrow[d, phantom, "\Leftarrow"] & {} \arrow[d, phantom, "\Leftarrow", near start] & \mathrm{lex} \arrow[r, tail] \arrow[dl, tail] \arrow[d, phantom, "\Leftarrow"] & \mathrm{prod} \arrow[dll, tail, bend left] \\
 & {} & \mathrm{skt} & {} &
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More Examples! (...and a sanity check)

- **First order doctrines**

$$\begin{tikzcd}
\mathrm{coh} \arrow[r] \arrow[drr, tail, bend right=10] & \mathrm{reg} \arrow[rr] \arrow[dr, tail] \arrow[l, phantom, shift right=4ex, "\Leftarrow"] & & \mathrm{lex} \arrow[r] \arrow[dl, tail] \arrow[ll, phantom, shift right=4ex, "\Leftarrow"] & \mathrm{prod} \arrow[dll, tail, bend left=10] \arrow[l, phantom, shift right=4ex, "\Leftarrow"] \\
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& & \arrow[u, phantom, "\Leftarrow"] & & \\
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- A model of a sketch $\mathcal{S}$ in a complete & cocomplete cat $\mathbb{C}$ is a functor sending each cone/cocone in $\mathcal{S}$ to lim/colim in $\mathbb{C}$.

[3] **Notation:** we call these **test** sketches.

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  $$\forall\, \mathcal{M} \in \mathbb{M} \qquad - \circ F\colon \mathsf{Skt}(\mathcal{S}, \mathcal{M}) \simeq \mathsf{Skt}(\mathcal{T}, \mathcal{M}).$$

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- Why is this useful? No `spoilers!`

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    $\mathsf{L}^m_{J_{\mathbb{S}}} :=$ min structure making $J_{\mathbb{S}}$ a limit-sketch morphism

Second Sanity Check: $\widehat{(-)}$ vs Topoi

- $(\mathbb{C}, J)$ site $\rightsquigarrow$ $\mathsf{Sh}(\mathbb{C})$ $=$ Classifying Topos of $(\mathbb{C}, J)$

$$\text{sites} \xrightarrow{\mathsf{Sh}} \text{Topoi}^{\text{op}}$$

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- $(\mathbb{C}, J)$ site $\rightsquigarrow$ $\mathsf{Sh}(\mathbb{C}) = $ Classifying Topos of $(\mathbb{C}, J)$
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    1. Underlying category is the same!
    2. No topos can see the difference (i.e. same models in any topos).
       $\rightsquigarrow$ component-wise is a Topoi-Morita equivalence

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Theorem (Di Liberti & L.)

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- A logos $\mathcal{S}$ is a *rounded* & *left* sketch.

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- $\mathcal{S}$ rounded $\Rightarrow$ $\widehat{\mathcal{S}}$ rounded

- A logos $\mathcal{S}$ is a *rounded* & *left* sketch.

E.g. Topoi

Where the Logoi at?

- $\mathcal{S} \in$ Skt is (left) rounded if $J_{\mathcal{S}}$ sends cones in $\mathrm{L}_{\mathcal{S}}$ to lim diagrams.

$$\begin{tikzcd} \mathbb{S} \arrow[r, "J_{\mathcal{S}}"] \arrow[d, "Y"'] & \widehat{\mathbb{S}} \\ \mathrm{Psh}(\mathbb{S}) \arrow[ur, "L"'] & \end{tikzcd}$$

E.g. prod, lex, reg, coh, Topoi

Proposition

- $\mathcal{S}$ rounded $\Leftrightarrow$ $\widehat{\mathcal{S}}$ is normal (i.e. $\mathrm{L}_{\mathcal{S}} \subseteq$ lim diag. & $\mathrm{C}_{\mathcal{S}} \subseteq$ colim diag.)
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- A logos $\mathcal{S}$ is a *rounded* & *left* sketch.

E.g. Topoi , λ-ary Topoi

Classifying Logoi

Theorem (Di Liberti & L.)

$\mathsf{Logoi}^{\mathsf{M}}$ is (bi)reflective in $r\mathsf{Skt}^{\mathsf{M}}$.
$$\begin{tikzcd} r\mathsf{Skt}^{\mathsf{M}} \arrow[r, bend right, "\mathcal{C}l[-]"'] \arrow[r, phantom, "\top"] & \mathsf{Logoi}^{\mathsf{M}} \arrow[l, bend right, "U"'] \end{tikzcd}$$

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$$\begin{tikzcd}
r\mathsf{Skt}^{\mathsf{M}} \arrow[r, hook] \arrow[d, bend left, "\mathcal{C}l[-]"] & \mathsf{Skt}^{\mathsf{M}} \arrow[d, bend left, "\widehat{(-)}"] \\
\mathsf{Logoi}^{\mathsf{M}} \arrow[r, hook] \arrow[u, dashed, bend left, "U"] \arrow[u, phantom, "\vdash"] & \mathsf{LSkt}^{\mathsf{M}} \arrow[u, dashed, bend left] \arrow[u, phantom, "\vdash"]
\end{tikzcd}$$

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$$\begin{tikzcd}
\mathsf{MSite} \arrow[r] \arrow[d, bend left, "\mathsf{Sh}"] \arrow[dr, phantom, "\Rightarrow"] & r\mathsf{Skt}^{\mathsf{M}} \arrow[r, hook] \arrow[d, bend left, "\mathcal{C}l[-]"] & \mathsf{Skt}^{\mathsf{M}} \arrow[d, bend left, "\widehat{(-)}"] \\
\mathsf{Topoi}^{\mathsf{op}} \arrow[r] \arrow[u, bend left, dashed, "J"] \arrow[u, phantom, "\vdash"] & \mathsf{Logoi}^{\mathsf{M}} \arrow[r, hook] \arrow[u, bend left, dashed, "U"] \arrow[u, phantom, "\vdash"] & \mathsf{LSkt}^{\mathsf{M}} \arrow[u, bend left, dashed] \arrow[u, phantom, "\vdash"]
\end{tikzcd}$$

Classifying Logoi

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$$\begin{tikzcd} \Phi\text{-ex} \arrow[r] \arrow[d, bend left, "P_\Phi"] \arrow[d, phantom, "\vdash"] & \mathsf{MSite} \arrow[r] \arrow[d, bend left, "\mathsf{Sh}"] \arrow[d, phantom, "\vdash"] & r\mathsf{Skt}^{\mathsf{M}} \arrow[r, hook] \arrow[d, bend left, "\mathcal{C}l[-]"] \arrow[d, phantom, "\vdash"] & \mathsf{Skt}^{\mathsf{M}} \arrow[d, bend left, "\widehat{(-)}"] \arrow[d, phantom, "\vdash"] \\ \infty\text{-}\mathsf{Ex}^{\mathsf{M}} \arrow[r] \arrow[u, dashed, bend left, "U"] & \mathsf{Topoi}^{\mathsf{op}} \arrow[r] \arrow[u, dashed, bend left, "J"] & \mathsf{Logoi}^{\mathsf{M}} \arrow[r, hook] \arrow[u, dashed, bend left, "U"] & \mathsf{LSkt}^{\mathsf{M}} \arrow[u, dashed, bend left] \end{tikzcd}$$

$\Rightarrow$

Wrap Up

Logic Fragment	Presentation	Classifying Object
Geometric	Site	Topos
Infinitary	Rounded Sketch	Logos

The End

Thanks for listening!